Identify fractions and decimals on a number line

To examine students' understanding of this core mathematics content, analyze their erroneous strategies.

By Meghan M. Shaughnessy

Open any upper elementary school mathematics textbook and you are likely to see tasks that ask students to label rational number points on a number line. A correct response to these tasks requires coordinating the directed distance of the marked point from zero in relation to a unit distance (the distance from zero to one or its equivalent) on the line. Then the relationship must be represented using one of two notational systems for rational number—fraction notation or decimal notation. For instance, the marked point in **figure 1a** can be labeled 1/5 because the interval from zero to one is divided into five parts of equal length (fifths) and the point is located at the end of the first of those parts. The marked point in **figure 1b** can be labeled 0.2 because the interval from zero to one is divided into ten parts of equal length (tenths) and the point is located at the end of the second of those parts.

These types of tasks are common not only in curricula in the upper elementary school grades but also on state assessments (e.g., California Department of Education 2009; Massachusetts Department of Elementary and Secondary Education 2008). Such tasks target foundational rational number concepts: A fraction (or a decimal) is more than a shaded part of an area, a part of a pizza, or a representation using base-ten blocks; a fraction (or a decimal) is also a number with a specific location on a number line. These concepts are described as core content in *Curriculum Focal Points* (NCTM 2006) and in the Common Core State Standards (2010). Further, a deep, flexible understanding of foundational fraction concepts provides students with a basis on which to judge whether computations with rational numbers make sense.

Upper elementary school teachers report that when such tasks are used to support students' understanding of these foundational fraction concepts, students produce incorrect answers, such as 2/6, 1/4, and 0.3. What do such answers and their underlying strategies reveal about the nature of students' understanding?

Understanding number lines

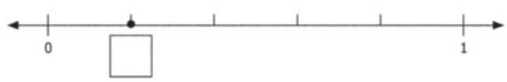
Representations of both the order and magnitude of numbers, number lines, and line segments are commonly used in early elementary school to represent whole numbers, and to represent both integers and fractions in the upper elementary grades. By middle school, students have typically been introduced to the Cartesian plane, which is constructed from two perpendicular number lines. Reflect for a moment on the mathematical properties and principles of the number line: What must elementary school students (and their teachers) understand?

If we look at any particular number on the number line, such as 1, numbers to its right are greater in value, and numbers to its left are less in value (see **fig. 2**). The number line also represents distance. The distance on the number line between zero and one (or its equivalent) is the unit distance (see **fig. 3a**), and the unit distance can be used to locate additional points on the

FIGURE 1

Labeling rational number points on a number line is a typical task in upper elementary school and on state assessments.

(a) What fraction should be written at the point?



(b) What decimal should be written at the point?

number line both inside and outside the bounds of a unit-distance interval. For example, once the distance between zero and one is defined, then the number 2 can be located by starting at the number 1 and moving to the right on the number line for a distance equal to the distance between zero and one (see **fig. 3b**). Once two numbers are marked on the number line, the location of all other numbers is fixed. If the distance between zero and one is defined, then the number 1/2 can be located by partitioning the interval from zero to one into two parts of equal length; the point at the end of the first part to the right of zero is one-half (see **fig. 3c**). If the distance between zero and one is defined, then the number 0.2 can be located by partitioning the interval from zero to one into ten parts of equal length; the point at the end of the second part to the right of zero is 0.2 (see **fig. 3d**).

Rather than giving your students a fraction or a decimal and asking them to mark it on a number line, have them use fraction or decimal notation to label points that have already been marked on the number line. One efficient strategy for labeling a point on the number line entails determining the number of parts of equal length that constitute the interval from zero to one and then determining the number of those parts that constitutes the interval from zero to the target point. In **figure 4**, ten parts of equal length constitute the interval from zero to one, and the unlabeled point is located at the end of two parts to the right of zero. This relationship can be represented using both fraction and decimal notation. When representing the relationship as a fraction, the number of parts of equal length that constitutes the interval from zero to one corresponds to the denominator, and the number of those parts that constitutes the interval from zero to the unlabeled point corresponds to the numerator. In other words, the unlabeled point can be labeled 2/10. When the number of equal parts that constitutes the interval from zero to one is a power of ten, the relationship can be represented implicitly via place value. In **figure 4**, the unlabeled point can be labeled 0.2, as it is two parts of length "one-tenth" to the right of zero. The location of the digit 2 represents "tenths."

Although these strategies for responding to tasks like those shown in **figure 1** and **figure 4** may seem straightforward to adults, such tasks pose challenges for upper elementary school students. Further, typical tasks used in classroom instruction may inadvertently mask students' understanding by not fully assessing the underlying content. For example, consider two number line tasks: In **figure 5a**, the interval from zero to one on the number line is not prepartitioned into parts of equal length, thus a correct response to this task must take into account that a point on the line is labeled by its directed distance from zero in relation to a unit distance. In contrast, the interval from zero to one on the number line in **figure 5b** is partitioned into parts of equal length,

FIGURE 2

Numbers increase in value as we move to the right on a number line, with a corresponding decrease in value as we move to the left.

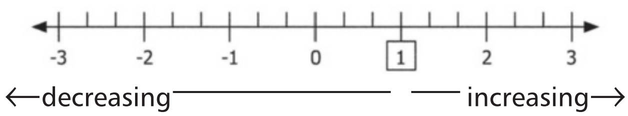
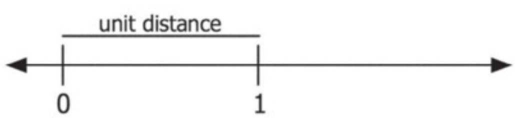


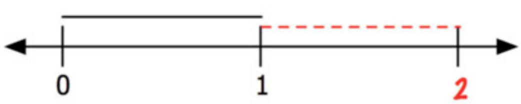
FIGURE 3

Number lines represent the order of numbers and their magnitudes.

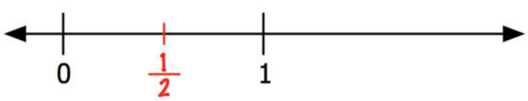
(a) Unit distance



(b) Locating an integer



(c) Locating a fraction



(d) Locating a decimal

so labels of 8/10 or 0.8 are unclear: Is the student considering distances on the number line or merely counting parts regardless of length?

Understanding incorrect answers

To explore the nature of students' difficulties when labeling rational number points on a number line, the author interviewed students in an urban school district in Northern California. The protocol she designed included a series of number line tasks asking students to label marked points on a number line as fractions and decimals (see **fig. 6**). For two of the number lines, she partitioned the intervals between consecutive integers into parts of equal length (see parts a–b). To problematize counts of parts on a number line, she partitioned the intervals between consecutive integers into unequal lengths (see parts c–d). Over the course of the interview, students were presented with each number line twice. First, students were asked, What fraction name would you call this point? Then they were asked, What decimal name would you call this point? For all tasks, students were instructed to write their fraction and decimal names and explain their answer verbally. (For a complete description of the methods used in this study, see Shaughnessy 2009.)

More students appropriately labeled points on the number line as decimals than as fractions, and more students appropriately labeled points when the interval from zero to one was equally partitioned than when the interval was unequally partitioned (see **fig. 7**). Both differences are statistically significant (Shaughnessy 2009). An analysis of students' incorrect answers and their verbal reasoning revealed four common error types characterizing' incorrect answers:

1. Using unconventional notation
2. Redefining the unit
3. A two-count strategy focusing on discrete tick marks (or parts) rather than distances
4. A one-count strategy focusing on discrete tick marks (or parts) rather than distances.

Using unconventional notation

Labeling a marked point on a number line with fraction or decimal notation requires an understanding of the conventions of these notational forms. In other words, although a student may coordinate the directed distance of the marked point from zero in relation to a unit distance, the representation the student uses to convey the relationship may not draw on standard conventions. For example, one student indicated that she would call the point 0.08 because it was "the eighth one" of ten total (see **fig. 8**). She coordinated the directed distance from zero to

FIGURE 4

The ten parts of equal length from zero to one can be represented with either a fraction or a decimal.

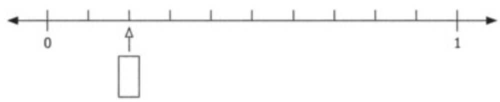


FIGURE 5

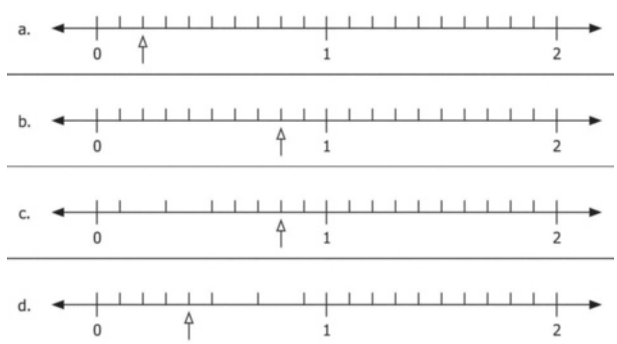
Typical classroom tasks can mask students' understanding.

(a) A nonstandard number line task assesses whether students are considering the directed distance of the point from zero in relation to the unit distance.

(b) In a standard number line task, when students label the point 8/10 (or 0.8), it is unclear whether they are considering distances on the number line or merely counting parts regardless of length.

FIGURE 6

To explore the nature of students' difficulties in labeling rational number points, the author interviewed thirty-one fifth graders. Students were to first identify the points on the number lines as fractions. The second time, they were to identify the points as decimals.



one (constituted by ten parts of equal length) with the distance from zero to the marked point (constituted by eight of those parts). However, 0.08 is an unconventional notation. Students in this study did not appear to use unconventional notation when asked to represent points on the line as fractions, but prior research has indicated that students' understanding of part-whole relationships (specifically, area models) and their use of conventional notation for fractions is independent (Saxe et al. 2005). It is plausible that younger students might have produced such responses as 10/8 when presented with the task shown in part c (see **fig. 6**). Like the response of 0.08, a response of 10/8 likely indicates a coordination of the directed distance of the marked point from zero in relation to a unit distance; however, the response indicates a limited understanding of conventional notation (see **fig. 9**).

FIGURE 7

The graph compares students' appropriately labeled points as decimals to appropriately labeled points as fractions.

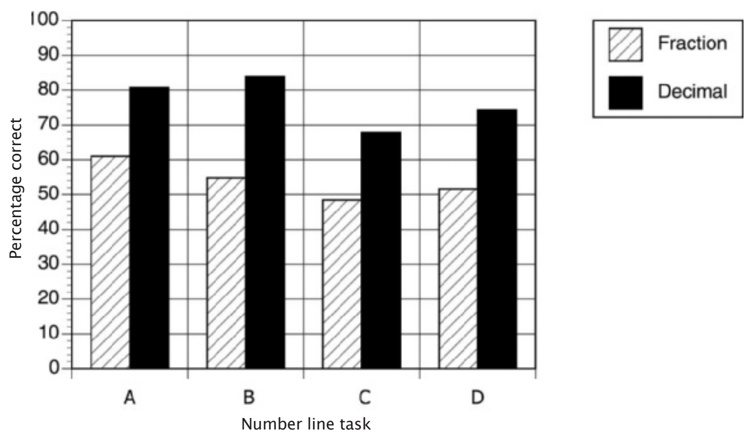


FIGURE 8

This student's decimal notation of 0.08 is unconventional.

FIGURE 9

Sometimes students redefine the unit, an error that is not apparent when the distances are equally partitioned for the student.

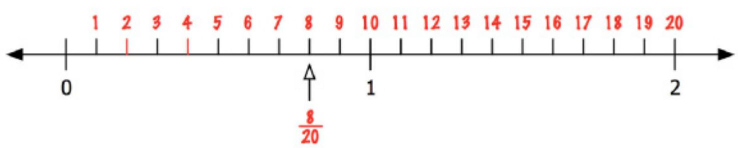
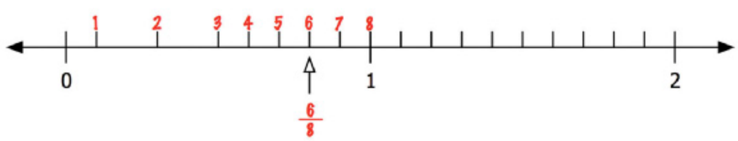


FIGURE 10

This two-count strategy erroneously focuses on discrete parts rather than distances.



Redefining the unit

During initial rational number instruction in the elementary school grades, students often see number lines marked from zero to one (see **fig. 1**), and thus the shown distance is the unit distance. When presented with number lines in which the shown distance is not the unit distance, such as the interview tasks (see **fig. 6**), students may redefine the unit distance on the number line and treat the entire distance shown as the unit distance rather than the distance between zero and one. For example, one student described the marked point as 8/20 (see **fig. 9**). This student added two tick marks to divide the interval between zero and one into ten parts of equal length and explained her answer in this way:

> It's ten from zero to one, so it has to be another ten from one to two. So that would make twenty; then I counted from zero to the arrow to get eight.

To determine the denominator, this student was attuned to the need to determine the number of parts of equal length that constitutes the distance, but she treated the entire length of the number line shown as the unit distance. In other words, her response reflected a redefining of the unit distance. In this study, students did not redefine unit distances when asked to label points as decimals.

A two-count strategy

Students often count discrete quantities, and incorrect answers on number line tasks may reflect a focus on discrete marks or parts without considering the directed distance of the point from zero in relationship to the unit distance. Students may use a two-count strategy for labeling a rational number point on the line as a fraction: One count, the denominator, is the count of the number of tick marks (or parts) in the unit interval. The second count, the numerator, is the number of tick marks (or parts) from zero to the target point. Note that distance is not being considered. For instance, one student represented the marked point as 6/8 (see **fig. 10**) because the interval between zero and one was divided into eight parts and the point is at the end of the sixth part. Thus, this student did not coordinate the distance of the marked point from zero with the unit distance. Instead, the student counted the number of parts into which the unit interval is divided to determine the denominator and counted the number of parts between zero and the target point to determine the numerator. In this particular case, this type of interpretation of the number is not visible on number lines in which intervals are equally partitioned for the student.

Dividing the unit interval into parts of equal length, however, is not necessarily an indicator that students are coordinating the directed distance of the marked point from zero with the unit distance. Students may partition the unit distance into equal parts but then count the number of tick marks rather than focus on the number of parts. For instance, one student partitioned the unit interval into equal parts (see **fig. 11**). Then, starting with zero and ending on one, she counted the number of tick marks (eleven) to determine the denominator. Finally, starting with zero and ending with the target point (nine), she counted the number of tick marks to determine the numerator.

Although such responses might be particularly visible on tasks in which the intervals between adjacent integers on the number lines given to students are unequally partitioned, this type of reasoning can also emerge on standard number line tasks, for instance, labeling the point in **figure 1a** as 2/6 because there are six total tick marks and the arrow points to the second one. For the tasks used in this study, this pattern of reasoning was specific to fractions.

FIGURE 11

To determine the numerator and denominator of a fraction, one student focused a two-count strategy on discrete tick marks rather than on distances.

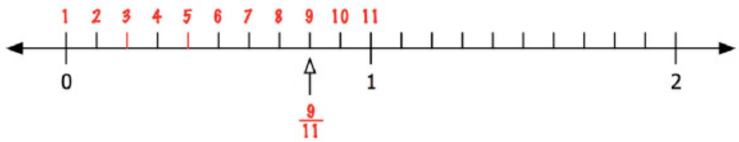
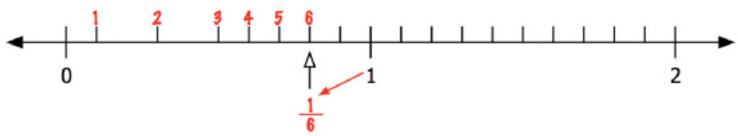


FIGURE 12

This student's one-count strategy for a fraction focused on the number of discrete tick marks, mistakenly used as the denominator.



A one-count strategy

Students may also count discrete quantities in ways that do not attempt to coordinate the number of discrete quantities from zero to the target point with the number of discrete quantities from zero to one. Students may use a one-count strategy focusing on the number of tick marks or parts. In the case of fractions, the number of tick marks, or parts (regardless of their length), from zero to the target point may be treated as the denominator. For example, one student labeled the point as 1/6 (see **fig. 12**). The student explained:

> One, two, three, four, five, six [*pointing to each tick mark to the right of "zero" and up to the target point*], and then I saw the one [*referring to the number 1 marked on the number line*] and put it there.

Thus, this particular student focused on the number of tick marks up to the target point (six), treated this number as the denominator, and then identified "1" as the numerator because "1" is marked on the number line to the right of the target point. Other errors categorized as a one-count strategy may involve slightly different reasoning. Some students may treat the number of tick marks up to the target point as the numerator, and zero as the denominator, because they are in the "zero space" (the space between zero and one).

In the case of decimals, students may treat each tick mark (or part) as a "tenth." For instance, starting with the first tick mark after zero, one student counted the number of tick marks up to the target point and labeled the point as 0.6 (see **fig. 13**). The student explained, "Zero, count six, and I got six." This type of reasoning was also used on tasks in which the distance from zero to one was partitioned into parts of equal length—students treated each tick mark from zero to one (including the tick mark labeled 0) as a tenth.

Instructional implications

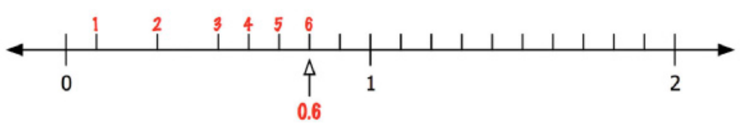
The findings of this study imply several points for teachers to consider. First, understanding the errors described can help make sense of the errors produced by students on standard number line tasks. For example, a student response of 2/6 to the task posed in **figure 1a** indicates that the student is likely using a two-count strategy focused on the number of tick marks rather than on the distances on the number line. These types of errors are not specific to the tasks shown in this article; rather they extend to other number line tasks in which students are asked to label rational number points on a number line.

Second, the types of errors described in this article may be useful prompts for specific tasks and discussions. Having students reflect on common misunderstandings can engage them in deepening their own understanding of the number line. They might discuss either their own errors or the responses of students "from another class."

Third, it is important for students to have experiences with a variety of number line tasks—number lines that are both prepartitioned and not partitioned, tasks in which students are given a fraction or a decimal and asked to locate it on the number line, and tasks in which students are asked to label a marked point using fraction and decimal notation.

FIGURE 13

A students' one-count strategy for a decimal focused on the number of discrete tick marks (assuming each was one-tenth).



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