Teaching Fractions: Rules and Reason

Many children cringe at the very thought of computing with fractions. What can the concerned teacher do to help?

Fortunately, many resources are available. Mathematics education textbooks, such as those by Cathcart et al. (2003) and by Reys et al. (2003), provide a useful overview of models and methods to help teachers. In addition, various studies offer more detailed descriptions to help the classroom teacher bring powerful fraction ideas into the classroom.

Riddle and Rodzwell (2000) found that young children have a beginning understanding of fraction concepts. In many instances, however, educators ignore this powerful intuitive knowledge, and as a result, students learn computation of fractions by rote. The authors conclude that although students must *eventually* understand the standard algorithms, "children should start by using their understanding of fractions to develop procedures that make sense to them" (p. 206).

Although young children have useful ideas about fractions, these intuitions do not automatically develop into a strong understanding of fraction concepts necessary for success in the upper elementary grades. In fact, Anderson, Anderson, and Wenzel (2000) note that many students have a poor understanding of fraction concepts, as well as an inability to recognize accurate visual representations of fractions. They describe a range of activities using oil and water to demonstrate the equivalence of fractions, as well as the addition and subtraction of fractions. As a result of these explorations, their students gained "a deeper understanding of fractions both visually and through hands-on activity" (p. 178).

We, too, believe in the importance of visual models and hands-on activity, and our approach emphasizes understanding rather than memorization. Rules, however, can be very useful to students and teachers. Our aim is to present rules that are both useful in the classroom and mathematically meaningful.

We first describe fraction equivalence, then the addition and subtraction of fractions. We present models for understanding the meaning of fractions and suggest ways to connect these models with the symbols for fractions. We also present a set of rules that have proved valuable in fourth and fifth grade, and we discuss how to make these rules meaningful to children.

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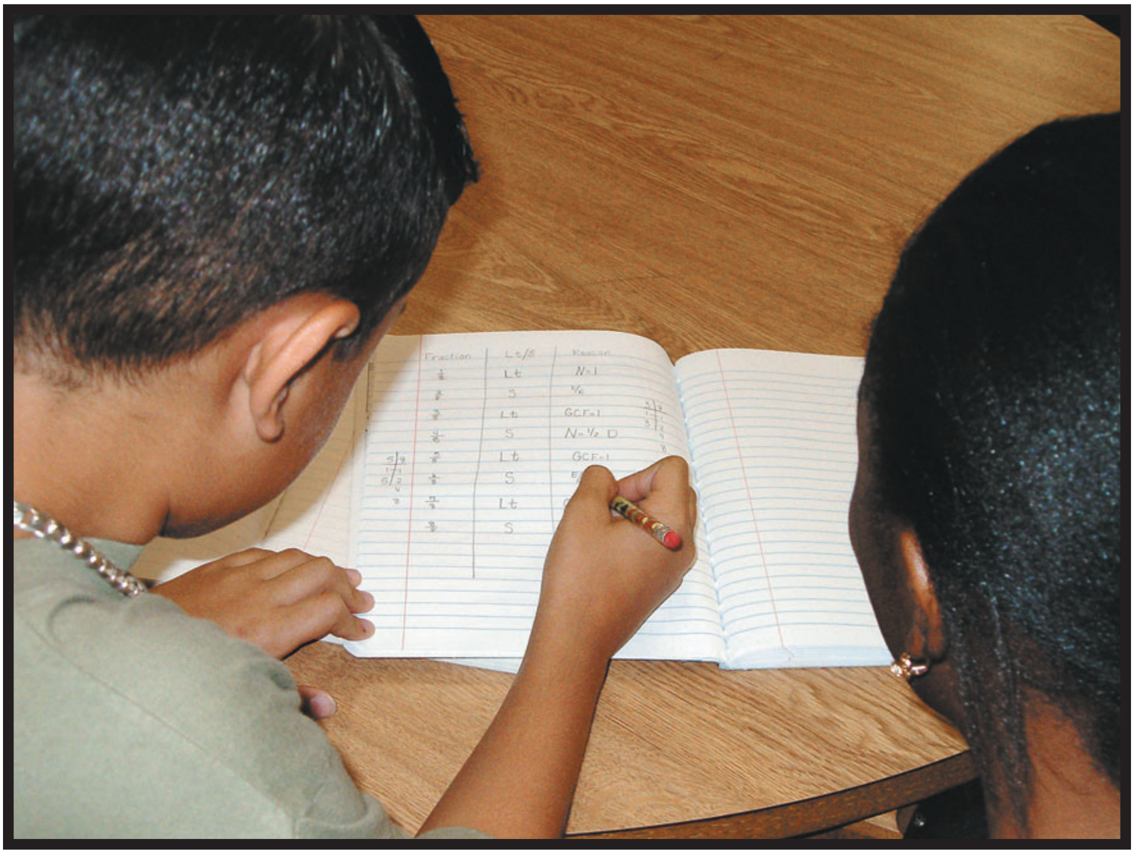
Fraction Equivalence

We begin with the fraction one-half. One can write other fractions that have the same value as one-half. We refer to these as *equivalent fractions*. For example, 1/2 = 2/4 = 3/6 = 4/8 = 5/10, and so on, forever.



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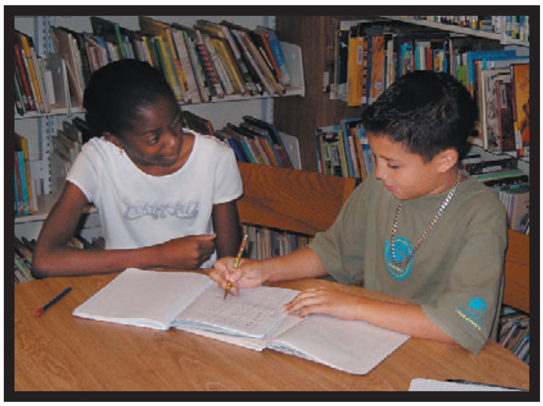
We have found that our students enjoy making this list (indeed, they extend it until at least 12/24) and then observing the pattern. For example, Sean noticed that the numerators increase by 1 and that the denominators increase by 2. Kathy noted that if you can cover one-half of the set, then the denominator must be even and the numerator must be half the denominator. Margaret noted that in each instance, the cross-products are equal. For example, 1/2 = 2/4 because $1 \times 4 = 2 \times 2$. Carlos gave another example of equivalent fractions: 3/6 = 5/10 because $3 \times 10 = 6 \times 5$.

After these discoveries have been discussed, we explain that among these forms, 1/2 is the simplest because the denominator is the smallest we could use and still write a fraction equivalent to 1/2. Consequently, we refer to 1/2 as being in *lowest terms* (or the simplest form), and we know that all other equivalent fractions (such as 2/4, and so on) can be simplified because they are not in lowest terms.

Rules for simplifying fractions

Because writing fractions in simplest form (or lowest terms) is extremely useful, we have developed a set of six simple rules that are easy to apply, along with a general rule that always works. Children can easily learn these rules and apply them successfully:



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The numerator-equals-the-denominator rule. If the numerator is equal to the denominator, the fraction can be simplified; it is equal to 1.

The numerator-of-1 rule. If the numerator is equal to 1 (and the denominator is greater than 1), the fraction is automatically in lowest, or simplest, terms. For example, the fraction 1/6 is in lowest, or simplest, terms because the numerator is 1.

The even-number rule. If both the numerator

and the denominator are even, the fraction can always be simplified. The fraction 2/6 can be simplified by dividing the numerator and the denominator by 2. We now have 1/3, which is in lowest terms. The even-number rule tells you that a fraction can be simplified; for example, 8/12 can be simplified to 4/6. Students must use the even-number rule again to simplify 4/6 to 2/3, which is in lowest terms.

The half rule. If the numerator is exactly one-half the denominator, the fraction can be simplified, and in lowest or simplest terms, it is equal to 1/2. For example, 3/6 = 1/2. The half rule is more powerful than the even-number rule. For example, the even-number rule tells us only that 12/24 can be simplified; the half rule tells us that it can be simplified to 1/2.

The consecutive-number rule. If the numerator and the denominator are consecutive numbers, the fraction is in lowest terms. We see that 5 and 6 are consecutive numbers, which tells us that 5/6 is in lowest terms.

The prime-denominator rule. If the denominator is a prime number, and the numerator is less than the denominator, the fraction is in lowest terms. For example, 2/5, 3/7, 8/11, and 10/13 are all in lowest terms.

In addition, we state a *general rule,* which uses the greatest common factor (GCF). If the GCF is 1, the fraction is in simplest form. If the GCF is greater than 1, the fraction can be simplified, and the simplest form is found by dividing each term by the GCF.

Summary

A fraction is in lowest terms if—

- the numerator is equal to 1;
- the numerator and denominator are consecutive numbers;
- the denominator is a prime number; or
- the GCF is 1.

A fraction can be simplified if—

- the numerator is equal to the denominator;
- the numerator is one-half the denominator;

- the numerator and denominator are both even numbers; or
- the GCF is greater than 1.

Students were fascinated by the rules, and they discovered that in many examples, one of the simple rules could be used to reduce a fraction to lowest terms. Robert and Carl noted two exceptions: 3/8 and 5/8. For these fractions, you must use the general rule. In both instances, the GCF is 1. Therefore, 3/8 and 5/8 are in lowest terms.

The general rule gives the same result as the simple rules. For example, for the fraction 8/12, the GCF is 4. When we divide numerator and denominator by 4, we see that 8/12 = 2/3. This result is the same as that achieved with the simple rules. In this instance, we need to apply the even-number rule twice (8/12 = 4/6 = 2/3). Then we must apply the consecutive-number rule to conclude that 2/3 is in lowest terms.

Addition of Fractions

When we consider the addition of fractions, we are looking for methods to give meaning to the process. Simply being able to apply a rule that gives the correct result is not enough. We need to know why the rule works, and to present a model that is easy to see and use.

Let us begin with a simple example: 1/2 + 1/3. We illustrate this example, using fraction circles, by putting one-half and one-third next to each other. The powerful idea is this: how can we cover the total region of 1/2 and 1/3 with the same size or kind of pieces, and how many of those pieces will we need to do so?

We can show that we can cover the one-half part with fourths, but we cannot cover the one-third part with fourths. We can cover the region one-half with sixths, however, and we can also cover the region one-third with sixths. We need three one-sixth pieces to cover the one-half region and two one-sixth pieces to cover the one-third region. Because the sixths are all the same size, we can count them to determine that five one-sixth pieces (that is, five-sixths) will be needed to cover the total region.

We can record this process with symbols:

1/2 + 1/3 =
3/6 + 2/6 = 5/6

In each instance, we can relate the symbols to the models. That is, we show that one-half is the same as three-sixths and that one-third is the same as two-sixths. Finally, when we combine three-sixths and two-sixths, we get five-sixths, which covers the combined region of one-half and one-third.

Students often make mistakes in fraction addition, so the teacher must describe and demonstrate the model clearly and show the connection with the symbols. The teacher should also monitor students' responses to ensure that they understand the concepts thoroughly.

Rules for adding fractions

Now that we have a clear model to demonstrate the addition of fractions, we can usefully introduce the rules. Nothing is wrong with having rules. The question is whether those rules are applied with understanding.

The rules for adding fractions can be stated as follows:

- First use a common denominator;
- then find the new numerators and add them.

Furthermore, we know that this process follows the model. In the example we have just explored, 1/2 + 1/3, we used sixths as the common denominator.

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The cross-product rule for adding fractions

After teaching the model and algorithm, the teacher needs to ensure that students understand the connection between the two. Once students understand this connection, they should solve several addition problems using the algorithm alone.

At this point, students are eager to learn a simpler way. Here, we introduce the cross-product method. Whereas the traditional algorithm requires paper and pencil, the cross-product method employs mental mathematics. It is easier and more efficient for children to use.

The cross-product method always gives the same result as the model and the algorithm do. For that reason, it offers a simpler way to add fractions with unlike

denominators. Consider the following example:

$$\frac{1}{2}+\frac{1}{3}$$

To find the sum of two fractions with unlike denominators, first cross multiply.

$$1 \times 3 = 3$$
$$2 \times 1 = 2$$

Then add the two cross-products: 3 + 2 = 5. The result becomes the new numerator. (In the model, it was the total number of sixths.) The new denominator is the product of the denominators: 2 × 3 = 6. (Sixths were the size of the pieces that covered both the one-half and the one-third region in the model.) Therefore, the sum is 5/6, just as found with the model and with the traditional algorithm.

Let us consider another example:

$$\frac{2}{3}+\frac{1}{4}$$

To find the sum of two fractions with unlike denominators, first cross multiply.

$$2 \times 4 = 8$$
$$3 \times 1 = 3$$

Next add the two cross-products: 8 + 3 = 11. The resulting value is the new numerator. The new denominator is the product of the denominators: 3 × 4 = 12. Therefore, the sum is 11/12.

Once children have mastered the models and the pencil-and-paper algorithm, they can use the cross-product method mentally. Indeed, we have found that students who had previously been frustrated by the calculations required by the curriculum have learned to use these methods successfully.

With the traditional algorithm, students must use paper and pencil; with the cross-product method, they use mental mathematics in a simpler and more efficient approach. Consider this example:

$$\frac{3}{4}+\frac{1}{6}$$

Children can readily calculate and add the cross-products: 3 × 6 = 18; 4 × 1 = 4; 18 + 4 = 22, which is the numerator. The denominator is 4 × 6 = 24. When children write the sum as 22/24, they immediately realize that the fraction can be simplified (on the basis of the even-number rule): 22/24 = 11/12, which they know is in lowest terms by the consecutive-number rule.

Subtraction of Fractions

Once students thoroughly understand addition of fractions, they can use our method to subtract fractions with unlike denominators.

Let us begin with a simple example: 1/2 – 1/3. We illustrate this example, using fraction circles, by putting a one-half region and a one-third region next to each other. The powerful idea is this: how can we cover the total region of 1/2 and 1/3 with the same size kind of pieces, and how many of those pieces will be needed? Because students already understand addition, they realize that they can cover each region with sixths.

Students realize that they need three-sixths to cover the one-half. We then show that removing two-sixths is the same as removing one-third. The result, the remaining one-sixth, is the answer to the problem.

We can show this procedure with symbols:

$$\frac{1}{2}-\frac{1}{3}=$$

$$\frac{3}{6}-\frac{2}{6}=\frac{1}{6}$$

The problem can also be solved using the cross-product method. In the instance of subtraction, the difference of the two cross-products is the new numerator and the product of the denominators is the new denominator.

At this point, we remind the students that subtraction is *not* commutative, and that order is important. To find the solution, we first find the minuend. We get this value by multiplying the top-left number (1) times the bottom-right number (3), yielding $1 \times 3 = 3$. To find the subtrahend, we multiply the bottom-left number (2) times the top-right number (1): $2 \times 1 = 2$. The result, $3 - 2 = 1$, is the numerator of the answer. To find the denominator, we multiply the denominators: $2 \times 3 = 6$. Therefore, the answer is 1/6.

At this point, we emphasize that the cross-product method gives the same answer as the common-denominator method and the model do. Two considerations are of utmost importance to teachers: (1) ensuring that students can perform each of the methods accurately and (2) helping students realize that all three methods give the same result.

After success with several examples, students see that the cross-product method gives the same answer as the common-denominator method and the model. For most fractions, students prefer the cross-product method. Once they are proficient at using the cross-product method, we give them more challenging problems. Consider the following example:

$$\frac{2}{3} - \frac{5}{8}$$

We first find the minuend: $2 \times 8 = 16$. We next find the subtrahend: $3 \times 5 = 15$. The numerator of the answer is $16 - 15 = 1$. The denominator is $3 \times 8 = 24$. Therefore, the answer is 1/24. The numerator-of-1 rule tells us that this fraction is in lowest terms.

Conclusion

Although dangers are inherent in using rules without meaning, teachers can introduce rules to enhance children's understanding of fractions. Our goal in this article is to make the rules as helpful and meaningful as possible. We hope this approach leads to more options for teachers. Mathematics instruction should foster number sense *and* abstraction. The rules and methods proposed in this article do both.

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