# Representations in Teaching and Learning Fractions

Fraction concepts continue to be one of the most challenging topics for elementary and middle school children (see, e.g., Kouba, Zawojewski, and Strutchens [1997]). One important factor in teaching and learning fractions is the use of representations. This article addresses four issues surrounding this topic: (1) tools for representing fractions, (2) methods of representing fractions, (3) fraction notations, and (4) fraction language.

## Tools for Representing Fractions

This article discusses fraction models. The terms *representation* and *model* are sometimes used almost interchangeably, but the two terms are not synonymous. For example, we use a *model* to *represent* a mathematical idea. The reason for this ambiguity is that both *representation* and *model* have several different meanings and share some meanings. A model can be a scale model of an actual object, a series of equations that mathematically model a physical phenomenon, a demonstration, something that illustrates or exemplifies a mathematical concept, concrete materials used in instruction, and so on. Similarly,

> the term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. . . . Moreover, the term applies to processes and products that are observable externally as well as to those that occur "internally," in the minds of people doing mathematics. All these meanings of representations are important to consider in school mathematics. (NCTM 2000, p. 67)

Even the forms mentioned can vary. In this article, I use the term *model* to refer to the instructional materials that we use, to distinguish it from the term *representation*.

We find three common fraction models in typical elementary and middle school mathematics textbooks; these are the linear model, the area model, and the discrete model (see **fig. 1**). Although other fraction models are used, these three seem to dominate school mathematics textbooks. In addition to static drawings like those on textbook pages, teachers and students often represent fractions using a variety of concrete objects. Most concrete materials are suitable for a particular fraction model, although some can be used for multiple types. For example, Cuisenaire rods are often used to model fractions linearly, whereas simple counters can be used as discrete models for fractions. Connecting cubes, in contrast, can be used as discrete models, just as simple counters are, or as linear models, by linking them to form "trains" of cubes.

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Pattern blocks are usually used in an area model of fractions. Although adults naturally use pattern blocks as area models of fractions, the concept as

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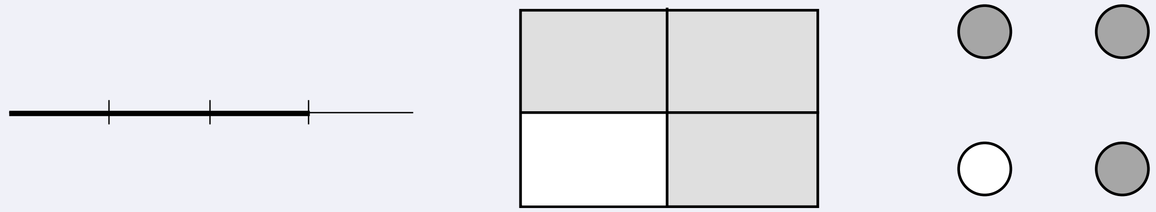
**FIGURE 1**

**Three different models for the fraction 3/4**



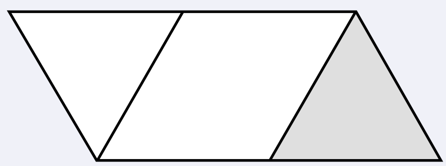
(a) Linear model

(b) Area model

(c) Discrete model

**FIGURE 2**

**Fraction 1/3 shown with pattern blocks**



implied by this choice of model is often left unexplained for children. Thus, when adults see a child using one blue and two green pattern blocks to show the fraction 1/3 (as shown in **fig. 2**), they often assume that the child does not understand fractions well enough to model them correctly. Nothing inherent in pattern blocks, however, prevents children from modeling fractions discretely by using each "piece" as the unit of counting. When we say that one-third of a class wears glasses, we clearly disregard such attributes as height, weight, and so on, of individual children in the classroom. What is different about pattern blocks that makes counting pieces inappropriate?

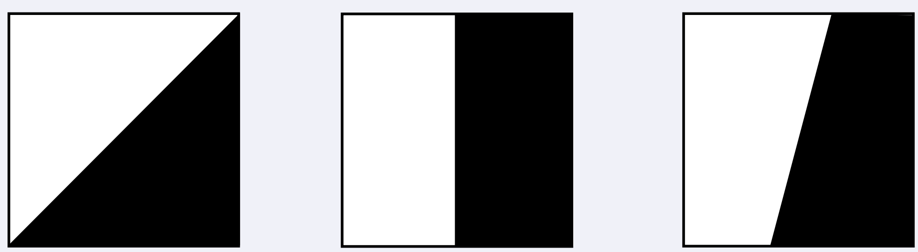
In considering the use of these three fraction models in instruction, we should ask, "Are they all appropriate for all students?" and "Are any of the models appropriate, or inappropriate, in some contexts?" Consider the following examples from research.

During an interview session, Kate, a bright fifth grader, was shown a square constructed with the seven pieces of a tangram set. She was then asked to determine what fraction of the square each tangram piece represented. She easily determined that the large triangle was 1/4 of the square. As she began looking at the other pieces, Kate decided that the medium triangle was 1/8 of the square because two of these triangles fit into a large triangle. She also concluded that the parallelogram was 1/7 of the square because it is one of seven pieces and that the tangram square was 1/9 of the larger square because she estimated that nine tangram squares would fit inside the large square. As she was explaining how she reached her conclusions, Kate realized that these three pieces (the medium triangle, the parallelogram, and the square) had the same area. She even demonstrated this fact by showing that each piece was equal to two small triangles. She was then asked whether her earlier conclusions made sense, that is, that three shapes of equal area represented different fractions of the same square. She was puzzled for a moment, but she replied, "I guess, since different shapes go differently into the square."

Ben was a second grader who participated in a different investigation (see Watanabe [1996] for more details). One of the questions that he was asked involved the three shapes shown in **figure 3.** He was first shown that two copies of each shape would make identical squares. He was then given one of each shape and asked to pretend that those were his favorite cookies, that he was really hungry, and that he could choose only one; which one

**FIGURE 3**

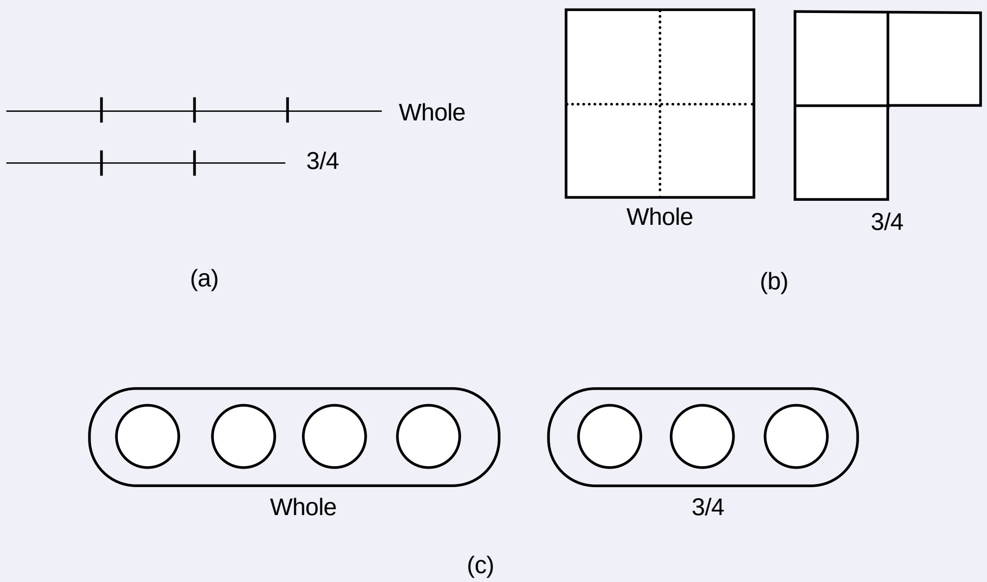
**Three shapes used in the cookie question**



(a) (b) (c)

FIGURE 4

Representing 3/4 using the ratio method




would he select? Ben chose the triangle shape because that was the largest. When asked how he knew that it was the largest, he was perplexed. He finally decided that the triangle was the largest because when he pretended to eat each piece, the triangle required the largest number of bites. When the same question was posed to sixteen fifth graders in individual interviews, fourteen selected one shape as the largest. Even after they were reminded of the initial demonstration (that two of each shape made identical squares), eight of these students still maintained that the shape they had selected was the largest.

What do these episodes tell us? One common theme is that these children's understanding of two-dimensional figures and their area measurements seems to affect their reasoning. Kate believed that different shapes could "fit" into a larger shape differently. Ben was puzzled by the fact that the perimeter did not appear to help him determine the largest cookie. His final solution would have worked had he realized that his "bites" needed to be of the same size. These episodes raise an important question: If children's understanding of two-dimensional shapes and their area measurement is still being developed, is the area model appropriate for discussing fractions with them?

## Methods of Representing Fractions

For each of the three models just discussed, we can use at least two distinct methods for representing fractions: (1) the part-whole method and (2) the comparison method. **Figure 1** shows the fraction 3/4 represented using the part-whole method with each model. The comparison method can be used with each model to represent the same fraction differently (see **fig. 4).** The main difference between these two methods is the relationship between the whole and the fractional part. In the part-whole method, the fractional part is embedded in the whole. In the comparison method, the whole and the fractional part are constructed separately. Also note that the comparison method represents a fraction by the relationship between the whole and fraction pieces. In other words, the fraction 3/4 is represented as in **figure 4** because the ratio of the fractional piece to the whole piece is 3 to 4 in the appropriate measurement or counting units. This method of representation reflects more of the meaning of fractions as ratios.

Some manipulatives lend themselves to one particular method of representation, whereas others can be used for both. For example, Cuisenaire rods, used as linear models, are more naturally suited for the comparison method than the part-whole method. The comparison method shows the whole and the fraction separately. When using Cuisenaire rods, we can say that one dark green rod will represent the whole. Then one red rod will be 1/3; one white, 1/6; one light green, 1/2; and so on. The part-whole method shows the fraction part embedded in the whole. By using Cuisenaire rods, we can make a "train" (put two rods end to end) of one red and one purple (making the total length 6 cm) with one red and one purple. Using the comparison method seems a more natural way to show the idea of fraction than the part-whole method does.

Connecting cubes, often used to represent fractions in the part-whole manner, can also be used to represent fractions in the comparison manner. When teachers use different types of manipulatives in their classrooms, they must pay attention to these two different methods of representing fractions. The way that fractions are represented with concrete materials must be consistent with the meanings of the fractions being explored. Thus, if fractions are introduced as parts of a whole, using connecting cubes may be more effective than using Cuisenaire rods. In addition, some mathematical ideas involving fractions may be better explored using another method of representation. For example, when children compare two fractions, representing fractions in the comparison method may be more beneficial in helping them understand the need for the common whole. Teachers may want to use Cuisenaire rods instead of connecting cubes for comparison activities. Moreover, children may benefit from more explicit treatment of the different ways that fractions can be represented.

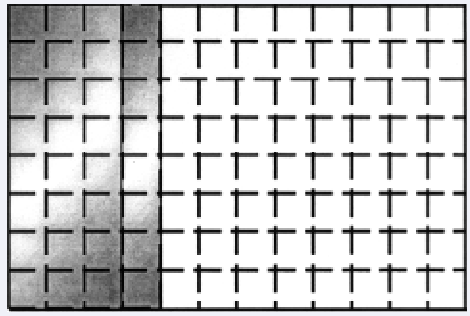
Another issue that should be considered is how appropriate these methods of representation are from a developmental perspective. For example, when we represent fractions using the part-whole method, the fractional part plays two different roles simultaneously: The part must be treated as an entity in itself *at the same time* that it is included in a larger entity. Thus, the part-whole method assumes children's ability to deal with the two nested quantities simultaneously, but this task is not easy for young children. This difficulty is similar to children's difficulty with the Piagetian class-inclusion tasks (Inhelder and Piaget 1964) and missing-addend problems (Steffe et al. 1983). If young children have difficulty with nested quantities, is the part-whole method of representing fractions developmentally appropriate?

This difficulty with the simultaneous roles of the fractional part may be compounded by the fact that concrete fraction models do not allow students to manipulate the part and whole independently. When a child moves the part away, the whole no longer exists. This situation may encourage some children to compare the part to the other part instead of the part to the whole. Consider the following example from research.

Jen was a fifth grader participating in the same study as Kate. She was shown the diagram in **figure 5** and asked whether this shape was 1/3 shaded. To answer this question, Jen first counted the number of squares that were inside the shaded region along the top side of the rectangle. She then counted the number of squares along the top that were not shaded. She decided that four shaded squares and eight unshaded squares along the top side of the large rectangle meant that the whole was 1/3 shaded. Jen seemed to have constructed a sophisticated understanding of fractions; her reasoning seemed to tell her that if the ratio of one part to the other part was 1 to 2, then the first part represented 1/3 of the whole. Jen appeared to be coordinating the part-whole perspective and the part-part perspective very well. We might ask, however, how she would deal with nonunit fractions. More important, should all students be able to make sense of such reasoning? If so, at what point in their study of fractions—when they are introduced to fractions or much later? How well will children be able to relate this reasoning with fraction symbolism? Teachers and curriculum developers must consider these questions.

FIGURE 5

Partially shaded figure shown to Jen



## Fraction Notations

Our conventional fraction notation seems simple to adults, but for young children, this symbolization is not obvious. For example, when a fourth grader was asked to show 2/3 using connecting cubes, she formed two groups of three cubes. Dismissing such an error as a child's misconception is easy; however, research indicates that for elementary school children, making sense of our fraction notation system is a complicated task.

For example, Casey, a second grader, was asked the balance question shown in **figure 6** (see Watanabe [1995] for more details). Casey was familiar

FIGURE 6

Balance question posed to second graders

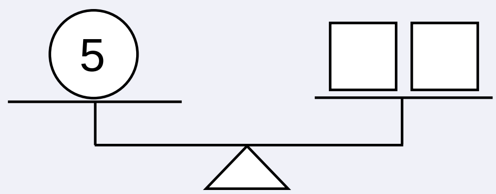
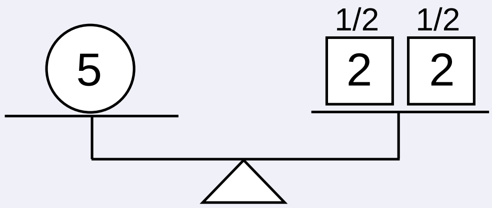


FIGURE 7

Casey's response to the balance question




with this type of problem, in which boxes of the same size and shape must contain the same number. She answered the problem as shown in **figure 7.** Then the following exchange took place:

*I.* Where does this "1" come from? Do you know?

*C.* [Shakes her head.]

*I.* OK . . .

*C.* Oh, yeah. *One* and a half. I guess it comes from *one.*

*I.* OK. So what if you had something like this? [Writes 1/4.] What do you think this means?

*C.* One and a fourth.

*I.* OK. Can you draw a picture of that?

*C.* No, I don't think . . . , well, I'll try. [Draws two semicircles.]

This exchange illustrates how challenging it is for young children to make sense of our fraction notations. The individual components in the fraction symbol 1/2 were almost like letters in a word for Casey. Although they looked familiar, they did not have any specific meaning by themselves. This lack of comprehension is not surprising if we recall the difficulty that children might have with nested quantities or their tendency to compare one part with the other part. From children's perspective, writing 1/3 as *1/2* (*1* for the part and *2* for the other part) seems to make much better sense. This representation, however, is not always useful. Just imagine multiplying two fractions represented in this manner, for example *1/3* × *1/1*, that is, 1/4 × 1/2. Although we do not want our students to learn to manipulate symbols mindlessly, we must also recognize that one useful feature of mathematical representations is the way they can be used to describe complex processes concisely. We must ask, therefore, "What role can such child-friendly notation systems play as children gain a more sophisticated understanding of fractions?"

## Fraction Language

In English, we use the combination of the counting number (numerator) and the ordinal number (denominator) to name, or verbally represent, fractions. The counting number signifies "how many" while the ordinal number signifies what is being counted. The use of a counting number in this system is natural and familiar to children. The ordinal number, however, is a completely different matter.

Other languages represent fractions in different ways. For example, in Japanese, the fraction 1/4 is read as *yon bun no ichi. Yon* is Japanese for "four" and *ichi* is "one." In other words, the denominator is read first. The word *bun* signifies partitioning, and *no* roughly corresponds with "of." Thus, the Japanese expression literally means "one of the four partitions." By reading the denominator first, along with using the word to emphasize partitioning, the Japanese fraction terminology seems to emphasize the act of partitioning much more strongly than English fraction words do. Expressing fractions in this manner is, perhaps, tied more closely to the part-whole interpretation of fractions.

How would such a difference in fraction language contribute to children's understanding of this concept, and is this question worth asking? Language differences, in a certain sense, are irrelevant because it is not feasible to teach English-speaking students Japanese first, then teach fractions, or vice versa. Understanding the different ways that other languages describe fractional quantities, however, may help us better understand the strengths and weaknesses of our particular ways of representing fraction concepts in words.

**The use of a counting number (numerator) is natural to children; an ordinal (denominator) is another matter**

The Japanese terminology's strength seems to be its more obvious connection to the multiplicative nature of fraction concepts. The way that it emphasizes the idea of partitioning is consistent with some researchers' recommendations that more emphasis be placed on children's actually partitioning wholes (see, e.g, Pothier and Sawada [1983]). This idea is not as explicit in English terminology and deserves more careful development in the classroom. One of the strengths of English terminology seems to be the connections to the more familiar counting world. The challenge is to help children develop understanding of the fractional units with which they are counting. In other words, we need to help children understand what is being counted.

This observation leads to another issue that is also related to fraction notations. In our conven-

FIGURE 8

Common error in showing 2/3 + 1/4 = 3/7 with connecting cubes

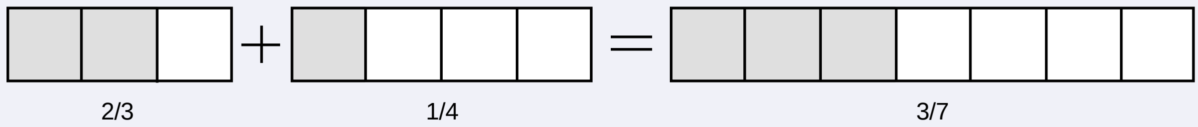



FIGURE 9

Showing 2/3 + 1/4 = 11/12 using the ratio method with connecting cubes

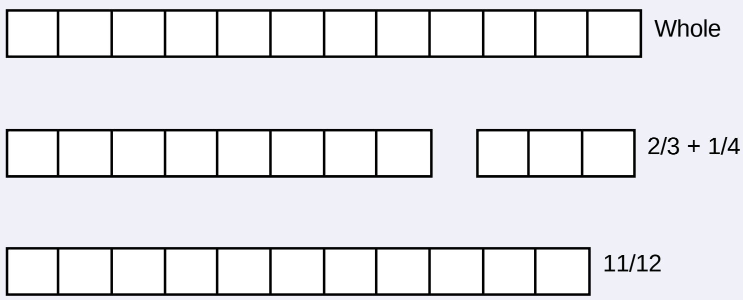
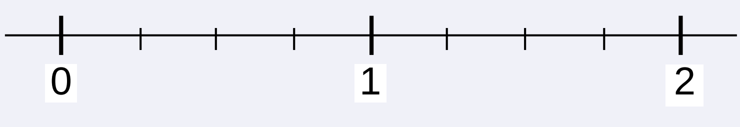



FIGURE 10

Many children have difficulty with tasks similar to this one.




tional fraction notation, the numerator and the denominator play different roles. The numerator counts while the denominator partitions the whole to create the unit to be counted. Introducing this standard notation in the primary grades simultaneously with fraction concepts may be too demanding for many students. Not only must they understand the concept of fractions, but they must also realize that familiar numerals are playing two different roles. Too often, both the numerators and the denominators are introduced as counting the number of parts in a fraction and the number of parts in a whole. For example, the 4 in 3/4 is often interpreted as the number of parts in the whole, but if the denominator is to signify what is being counted, then 4 cannot be the number of parts in the whole. Rather, 4 in the denominator should signify the fractional unit that is obtained when the whole is partitioned into four parts.

Gunderson and Gunderson (1957) suggested that at the beginning of fraction instruction, we should write out the fraction words rather than use standard notations. For example, we should write *3-fourths*, instead of 3/4. This suggestion seems to be consistent with the need for more emphasis on fraction units.

## Fractions as Numbers

As children continue with their learning of fractions, at some point, they must develop the notion of fractions as numbers. Such ideas as equivalent fractions and arithmetic operations will not make much sense unless fractions are understood as numbers, or quantities. Teachers must pay close attention to the way that fractions are represented, because these representations can influence children's problem-solving strategies. When adding 2/3 and 1/4, for example, a common mistake is to represent each fraction as shown in **figure 8** and put them together to form 3/7 as the answer. As mentioned previously, when these fractions are represented with the part-whole method, the fractional parts cannot be manipulated independently from their wholes. Thus, as children attempt to combine, or add, them, putting together both the fractional parts and the wholes seems reasonable.

This situation illustrates that the part-whole method of representing a fraction does not emphasize sufficiently the idea that a fraction is a single quantity. It represents a relationship between the part and the whole, and one without the other does not mean much in this representation. Representing fractions using the comparison method may be more useful in this context (see **fig. 9).** Although this representation also signifies the relationship between the part and the whole, the fraction can be manipulated independently. However, this approach may still not adequately address the need for children to understand fractions as numbers.

Another common tool for representing fractions is the number line. Because number lines are often used in primary grades while children are investigating whole numbers, some teachers may think that number lines are useful tools to teach children relationships between whole numbers and fractions. Research has consistently shown, however, that students have difficulty using number lines to work with fractions. For example, students' difficulty with problems like the one shown in **figure 10** has been well documented in a variety of contexts (see, e.g., Larson [1980]). We may find that number lines do not help students develop a sense of fractions as numbers but that number-line representations make sense only to those students who already understand fractions as numbers.

## Implications for Teaching and Learning Fractions

Some may argue that we are paying too much attention to details, but I believe that representation is a complex issue. As we think about the role of representation in teaching and learning mathematics in general, and fractions in particular, we need to remember that representations do not embody mathematical meanings independently. Representations are meaningful to the one who created them, whether that creator is the teacher or the students. The teacher's responsibility is to make sense of students' representations and to make sure that the tools, methods, notation, and language of representation used in the curriculum are developmentally and mathematically appropriate for the students. That responsibility can be met only when we pay close attention to these details.

Finally, remember that this article presents an overview of representation. We can examine many other issues involving representations in general and representations of fractions in particular, for example, the effects of virtual representations. Interested readers are encouraged to reflect further on these and other issues.

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